GEOMETRICAL DESCRIPTION OF SMOOTH PROJECTIVE SYMMETRIC VARIETIES WITH PICARD NUMBER ONE

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Abstract

In [Ru2] we have classified the smooth projective symmetric G-varieties with Picard number one (and G semisimple). In this work we give a geometrical description of such varieties. In particular, we determine their group of automorphisms. When this group, $Aut(X)$, acts non-transitively on X, we describe a G-equivariant embedding of the variety X in a homogeneous variety (with respect to a larger group).

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A Gorenstein normal algebraic variety X over $\mathbb{C}$ is called a Fano variety if the anticanonical divisor is ample. The Fano surfaces are classically called Del Pezzo surfaces. The importance of Fano varieties in the theory of higher dimensional varieties is similar to the importance of Del Pezzo surfaces in the theory of surfaces. Moreover Mori's program predicts that every uniruled variety is birational to a fiberspace whose general fiber is a Fano variety (with terminal singularities).

Often it is useful to subdivide the Fano varieties in two kinds: the Fano varieties with Picard number equal to one and the Fano varieties whose Picard number is strictly greater of one. For example, there are many results which give an explicit bound to some numerical invariants of a Fano variety (depending on the Picard number and on the dimension of the variety). Often there is an explicit expression for the Fano varieties of Picard number equal to one and another expression for the remaining Fano varieties.

We are mainly interested in the smooth projective spherical varieties with Picard number one. The smooth toric (resp. homogeneous) projective varieties with Picard number one are just projective spaces (resp. G/P with G simple and P maximal). B. Pasquier has recently classified the smooth projective horospherical varieties with Picard number one (see [P]). In a previous work we have classified the smooth projective symmetric G-varieties with Picard number one and G semisimple (see [Ru2]). One can easily show that they are all Fano, because the canonical bundle cannot be ample. We have also obtained a partial classification of the smooth Fano complete symmetric varieties with Picard number strictly greater of one (see [Ru1]). Our classification of the smooth

projective symmetric varieties with Picard number one is a combinatorial one, so we are naturally interested to give a geometrical description of such varieties. In particular, we have proved that, given a symmmetric space G/H, there is at most a smooth completion X of G/H with Picard number one and X must be projective (see [Ru2], Theorem 3.1). We will prove that the automorphism group of a such variety X can act non-transitively on X only if the rank of X is 2. It would be interesting to find a reason for such exceptionality of the rank 2 case. Unfortunately, our prove does not explain completely this fact, because there is a part of the proof that it is a case-to-case analysis. The homogeneousness of the rank one varieties was proved first by Ahiezer in [A1].

More precisely we have proved that:

Theorem 1 *Let X be a smooth projective completion of a symmetric space G/H with Picard number one (where G is semisimple and simply connected). Then $Aut(X)$ does not act transitively on X if and only if: i) the restricted root system has type either A_2 or G_2 and ii) the subgroup H is the subgroup of invariants G^θ.*

There are six varieties which are not homogenous; their connected automorphism group $Aut^0(X)$ is isomorphic to G up to isogeny. These varieties can be realized as intersection of hyperplane sections of a homogeneous projective variety with Picard number one. Moreover, they are someway related to the exceptional groups; in particular there are two varieties related to G_2 and four varieties related to the magic square of Freudenthal. The non-homogeneous varieties with restricted root system of type A_2 are obtainable as hyperplane sections of the Legendrian varieties in the third row of the Freudenthal magic square. Moreover each one is contained in the the others with bigger dimension. The completion of SL_3 was already studied by J. Buczyński (see [Bu]).

More precisely we will prove the following theorems:

Theorem 2 *Let G/G^θ be a symmetric space whose restricted root system has type G_2. We have the following possibilities for the smooth completion of G/H with Picard number one:*

1. *the smooth equivariant completion with Picard number one of the symmetric variety $G_2/(SL_2 \times SL_2)$ of type G. In this case, $Aut(X) = G_2$. The involution θ can be extended to an involution of SO_7, whose invariant subgroup is $S(O_3 \times SO_4)$. The unique equivariant smooth completion of $SO_7/N_{SO_7}(S(O_3 \times SO_4))$ with Picard number one is isomorphic to $\mathbb{G}_3(7) \subset \mathbb{P}^{34}$ and X is the intersection of $\mathbb{G}_3(7)$ with a 27-dimensional subspace of $\mathbb{P}^{34}$. If we interprete $\mathbb{C}^7$ as the subspace of imaginary elements of the complexified octonions $\mathbb{O}$, then X parametrizes the subspaces W of $\mathbb{C}^7$ such that $W \oplus \mathbb{C}1$ is a subalgebra of $\mathbb{O}$ isomorphic to the complexified quaternions.*

2. *the smooth equivariant completion with Picard number one of the symmetric $(G_2 \times G_2)$-variety G_2. In this case, $Aut(X)$ is generated by $G_2 \times G_2$ and*

θ. The involution θ can be extended to an involution of $SO_7 \times SO_7$, with invariant subgroup equal to the diagonal. The unique equivariant smooth completion of SO_7 with Picard number one is isomorphic to $\mathbb{IG}_7(14) \subset \mathbb{P}^{63}$ and X is the intersection of $\mathbb{IG}_7(14)$ with a 49-dimensional subspace of $\mathbb{P}^{63}$.

Theorem 3 *Let G/G^θ be a symmetric space whose restricted root system has type A_2. We have the following possibilities for the smooth completion of G/H with Picard number one:*

1. *the smooth equivariant completion with Picard number one of the symmetric variety SL_3/SO_3 of type AI; it is an hyperplane section of $\mathbb{LG}_3(6)$.*
2. *the smooth equivariant completion with Picard number one of the symmetric variety SL_3; it is an hyperplane section of $\mathbb{G}_3(6)$.*
3. *the smooth equivariant completion with Picard number one of the symmetric variety SL_6/Sp_6 of type AII; it is an hyperplane section of $\mathbb{S}_{12}$.*
4. *the smooth equivariant completion with Picard number one of the symmetric variety E_6/F_4 of type EII; it is an hyperplane section of $E_7/P_7 \equiv \mathbb{G}_3(\mathbb{O}^6)$.*

Moreover, $SL_3/SO_3 \subset SL_3 \subset SL_6/Sp_6 \subset E_6/F_4$ and also their smooth completions with Picard number one are contained nested in each other:

$$\begin{tikzcd}
\overline{SL_3/SO_3} \arrow[r, hook] \arrow[d, hook] & \overline{SL_3} \arrow[r, hook] \arrow[d, hook] & \overline{SL_6/Sp_6} \arrow[r, hook] \arrow[d, hook] & \overline{E_6/F_4} \arrow[d, hook] \\
\mathbb{LG}_3(6) \arrow[r, hook] & \mathbb{G}_3(6) \arrow[r, hook] & \mathbb{S}_{12} \arrow[r, hook] & \mathbb{G}_3(\mathbb{O}^6)
\end{tikzcd}$$

If G/H is different from SL_3, then the automorphism group of X is generated by $Aut^0(X)$ and θ. If, instead, $G/H = SL_3$ then $Aut^0(X)$ has index four in $Aut(X)$.

We give also an explicit description of the smooth projective symmetric varieties with Picard number one over which $Aut(X)$ acts transitively: in particular, we will describe their connected automorphism group and the immersion of G/H in $Aut^0(X)/Stab_{Aut^0(X)}(H/H)$.

In the first section we introduce the notation and recall some general facts about symmetric varieties. In the second one we cite a theorem of D. Ahiezer about the homogeneousness of the rank one varieties. In the third one we study the varieties of rank two which does not belong to an infinite family; in particular we consider all the varieties which are not homogeneous. In the forth section we study the remaining varieties; in particular we will study all the varieties of rank at least three.

1 Introduction and notations

1.1 Symmetric varieties and colored fans

In this section we introduce the necessary notations. The reader interested to the embedding theory of spherical varieties can see [Br3] or [T2]. In [V1] is explained such theory in the particular case of the symmetric varieties.

1.1.1 First definitions

Let G be a connected semisimple algebraic group over $\mathbb{C}$ and let θ be an involution of G. Let H be a closed subgroup of G such that $G^\theta \subset H \subset N_G(G^\theta)$. We say that G/H is a symmetric homogeneous variety. An equivariant embedding of G/H is the data of a G-variety X together with an equivariant open immersion $G/H \hookrightarrow X$. A normal G-variety is called a spherical variety if it contains a dense orbit under the action of an arbitrarily chosen Borel subgroup of G. One can show that an equivariant embedding of G/H is a spherical variety if and only if it is normal (see [dCP1], Proposition 1.3). In this case we say that it is a symmetric variety. We say that a subtorus of G is split if $\theta(t) = t^{-1}$ for all its elements t. We say that a split torus of G of maximal dimension is a maximal split torus and that a maximal torus containing a maximal split torus is maximally split. One can prove that any maximally split torus is θ stable (see [T2], Lemma 26.5). We fix arbitrarily a maximal split torus T^1 and a maximally split torus T containing T^1. Let R_G be the root system of G with respect to T and let R_G^0 be the subroot system composed by the roots fixed by θ. We set $R_G^1 = R_G \setminus R_G^0$. We can choose a Borel subgroup B containing T such that, if α is a positive root in R_G^1, then $\theta(\alpha)$ is negative (see [dCP1], Lemma 1.2). One can prove that BH is dense in G (see [dCP1], Proposition 1.3).

1.1.2 Colored fans

Now, we want to define the colored fan associated to a symmetric variety. Let $D(G/H)$ be the set of B-stable prime divisors of G/H; its elements are called colors. Since BH/H is an affine open set, the colors are the irreducible components of $(G/H) \setminus (BH/H)$. We say that a spherical variety is simple if it contains one closed orbit. Let X be a simple symmetric variety with closed orbit Y. Let $D(X)$ be the subset of $D(G/H)$ consisting of the colors whose closure in X contains Y. We say that $D(X)$ is the set of colors of X. To each prime divisor D of X, we can associate the normalized discrete valuation v_D of $\mathbb{C}(G/H)$ whose ring is the local ring $\mathcal{O}_{X,D}$. One can prove that D is G-stable if and only if v_D is G-invariant, i.e. $v_D(s \cdot f) = v_D(f)$ for each $s \in G$ and $f \in \mathbb{C}(G/H)$. Let N be the set of all G-invariant valuations of $\mathbb{C}(G/H)$ taking value in $\mathbb{Z}$ and let $N(X)$ be the set of the valuations associated to the G-stable prime divisors of X. Observe that each irreducible component of $X \setminus (G/H)$ has codimension one, because G/H is affine. Let $S := T/T \cap H \simeq T \cdot x_0$,

where $x_0 = H/H$ denotes the base point of G/H. One can show that the group $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is isomorphic to the character group $\chi(S)$ of S (see [V1], §2.3); in particular, it is a free abelian group. We define the rank of G/H as the rank of $\chi(S)$. We can identify the dual group $Hom_{\mathbb{Z}}(\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*, \mathbb{Z})$ with the group $\chi_*(S)$ of one-parameter subgroups of S; so we can identify $\chi_*(S) \otimes \mathbb{R}$ with $Hom_{\mathbb{Z}}(\chi(S), \mathbb{R})$. The restriction map to $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is injective over N (see [Br3], §3.1 Corollaire 3), so we can identify N with a subset of $\chi_*(S)\otimes\mathbb{R}$. We say that N is the valuation semigroup of G/H. Indeed, N is the semigroup constituted by the vectors in the intersection of the lattice $\chi_*(S)$ with an appropriate rational polyhedral convex cone $\mathcal{C}N$, called the valuation cone. For each color D, we define $\rho(D)$ as the restriction of v_D to $\chi(S)$. In general, the map $\rho : D(G/H) \to \chi_*(S) \otimes \mathbb{R}$ is not injective.

Let $C(X)$ be the cone in $\chi_*(S) \otimes \mathbb{R}$ generated by $N(X)$ and $\rho(D(X))$. We denote by $cone(v_1, ..., v_r)$ the cone generated by the vectors $v_1, ..., v_r$. Given a cone C in $\chi_*(S)\otimes\mathbb{R}$ and a subset D of $D(G/H)$, we say that (C, D) is a colored cone if: 1) C is generated by $\rho(D)$ and by a finite number of vectors in N; 2) the relative interior of C intersects $\mathcal{C}N$. The map $X \to (C(X), D(X))$ is a bijection from the set of simple symmetric varieties to the set of colored cones (see [Br3], §3.3 Théorème).

Given a symmetric variety $\widetilde{X}$ (not necessarily simple), let $\{Y_i\}_{i\in I}$ be the set of G-orbits. Observe that $\widetilde{X}$ contains a finite number of G-orbits, thus $\widetilde{X}_i := \{x \in \widetilde{X} \mid \overline{G \cdot x} \supset Y_i\}$ is open in $\widetilde{X}$ and is a simple symmetric variety whose closed orbit is Y_i. We define $D(\widetilde{X})$ as the set $\bigcup_{i\in I} D(\widetilde{X}_i)$. The family $\{(C(\widetilde{X}_i), D(\widetilde{X}_i))\}_{i\in I}$ is called the colored fan of $\widetilde{X}$ and determines completely $\widetilde{X}$ (see [Br3], §3.4 Théorème 1). Moreover $\widetilde{X}$ is complete if and only if $\mathcal{C}N \subset \bigcup_{i\in I} C(\widetilde{X}_i)$ (see [Br3], §3.4 Théoremè 2).

1.1.3 Restricted root system

To describe the sets N and $\rho(D(G/H))$, we need to associate a root system to G/H. The subgroup $\chi(S)$ of $\chi(T^1)$ has finite index, so we can identify $\chi(T^1) \otimes \mathbb{R}$ with $\chi(S) \otimes \mathbb{R}$. Because T is θ-stable, θ induces an involution of $\chi(T) \otimes \mathbb{R}$ which we call again θ. The inclusion $T^1 \subset T$ induces an isomorphism of $\chi(T^1) \otimes \mathbb{R}$ with the (-1)-eigenspace of $\chi(T) \otimes \mathbb{R}$ under the action of θ (see [T2], §26). Denote by W_G the Weyl group of G (with respect to T) and let $(\cdot, \cdot)$ be the Killing form over $span_{\mathbb{R}}(R_G)$. We denote with the same symbol the restriction of $(\cdot, \cdot)$ to $\chi(T^1) \otimes \mathbb{R}$, thus we can identify $\chi(T^1) \otimes \mathbb{R}$ with its dual $\chi_*(T^1) \otimes \mathbb{R}$. The set $R_{G,\theta} := \{\beta - \theta(\beta) \mid \beta \in R_G\} \setminus \{0\}$ is a root system in $\chi(S)\otimes\mathbb{R}$ (see [V1], §2.3 Lemme), which we call the restricted root system of (G, θ); we call the non zero $\beta - \theta(\beta)$ the restricted roots. Usually we denote by β (respectively by α) a root of R_G (respectively of $R_{G,\theta}$); often we denote by ϖ (respectively by ω) a weight of R_G (respectively of $R_{G,\theta}$). In particular, we denote by $\varpi_1, ..., \varpi_n$ the fundamental weight of R_G (we have chosen the basis of R_G associated to B). Notice however that the weights of $R_{G,\theta}$ are weights

of R_G. The involution $\iota := -\varpi_0 \cdot \theta$ of $\chi(T)$ preserves the set of simple roots; moreover ι coincides with $-\theta$ modulo the lattice generated by R^0_G (see [T2], p. 169). Here ϖ_0 is the longest element of the Weyl group of R^0_G. We denote by $\alpha_1, ..., \alpha_s$ the elements of the basis $\{\beta - \theta(\beta) \,|\, \beta \in R_G \text{ simple}\} \setminus \{0\}$ of $R_{G,\theta}$. Let b_i be equal to $\frac{1}{2}$ if $2\alpha_i$ belongs to $R_{G,\theta}$ and equal to one otherwise; for each i we define $\alpha_i^\vee$ as the coroot $\frac{2b_i}{(\alpha_i,\alpha_i)}\alpha_i$. The set $\{\alpha_1^\vee, ..., \alpha_s^\vee\}$ is a basis of the dual root system $R^\vee_{G,\theta}$, namely the root system composed by the coroots of the restricted roots. We call the elements of $R^\vee_{G,\theta}$ the restricted coroots. Let $\omega_1, ..., \omega_s$ be the fundamental weights of $R_{G,\theta}$ with respect to $\{\alpha_1, ..., \alpha_s\}$ and let $\omega_1^\vee, ..., \omega_s^\vee$ be the fundamental weights of $R^\vee_{G,\theta}$ with respect to $\{\alpha_1^\vee, ..., \alpha_s^\vee\}$. Let C^+ be the positive closed Weyl chamber of $\chi(S) \otimes \mathbb{R}$; we call $-C^+$ the negative Weyl chamber. Given a dominant weight λ of G, we denote by $V(\lambda)$ the irreducible representation of highest weight λ.

We want to give a description of the weight lattice of $R_{G,\theta}$. We say that a dominant weight $\varpi \in \chi(T)$ is a spherical weight if $V(\varpi)$ contains a non-zero vector fixed by G^θ. In this case, V^{G^θ} is one-dimensional and $\theta(\varpi) = -\varpi$. Thus, we can think ϖ as a vector in $\chi(S) \otimes \mathbb{R}$. One can show that the lattice generated by the spherical weights coincides with the weight lattice of $R_{G,\theta}$. See [CM], Theorem 2.3 or [T2], Proposition 26.4 for an explicit description of the spherical weights. Such description implies that the set of dominant weights of $R_{G,\theta}$ is the set of spherical weights and that C^+ is the intersection of $\chi(S) \otimes \mathbb{R}$ with the positive closed Weyl chamber of the root system R_G.

1.1.4 The sets N and $D(G/H)$

The set N is equal to $-C^+ \cap \chi_*(S)$; in particular, it consists of the lattice vectors of the rational polyhedral convex cone $\mathcal{C}N = -C^+$. The set $\rho(D(G/H))$ is equal to $\{\alpha_1^\vee, ..., \alpha_s^\vee\}$ (see [V1], §2.4, Proposition 1 and Proposition 2). For each i, the fibre $\rho^{-1}(\alpha_i^\vee)$ contains at most 2 colors (see [V1], §2.4, Proposition 1). We say that a simple restricted root α_{i_0} is exceptional if there are two distinct simple roots β_{i_1} and β_{i_2} in R_G such that: 1) $\beta_{i_1} - \theta(\beta_{i_1}) = \beta_{i_2} - \theta(\beta_{i_2}) = \alpha_{i_0}$; 2) either $\theta(\beta_{i_1}) \neq -\beta_{i_2}$ or $\theta(\beta_{i_1}) = -\beta_{i_2}$ and $(\beta_{i_1}, \beta_{i_2}) \neq 0$. In this case we say that also $\alpha_{i_0}^\vee$, θ and all the equivariant embeddings of G/H are exceptional. If G/H is exceptional, then ρ is not injective. We say that G/H contains a Hermitian factor if the center of $[G,G]^\theta$ has positive dimension. If G/H does not contain a Hermitian factor, then ρ is injective (see [V1], §2.4, Proposition 1). If ρ is injective, we denote by $D_{\alpha^\vee}$ the unique color contained in $\rho^{-1}(\alpha^\vee)$.

1.1.5 Local description of symmetric varieties

Let X be a symmetric variety and let Y be a closed orbit of X. One can show that there is a unique B-stable affine open set X_B that intersects Y and is minimal for this property. Moreover, the complement $X \setminus X_B$ is the union of the B-stable prime divisors not containing Y (see [Br3], §2.2, Proposition). The stabilizer P of X_B is a parabolic subgroup containing B. Suppose X smooth

or non-exceptional and let L be the standard Levi subgroup of P. Then L is θ-stable, $\theta(P) = P^-$ and the P-variety X_B is the product $R_u P \times Z$ of the unipotent radical of P and of an affine L-symmetric variety Z containing x_0 (see [Br3], §2.3, Théorème and [Ru2], Lemma 2.1). If Y is projective, then Z contains a L-fixed point whose stabilizer in G is P^-. Given a root β, let U_β be the unipotent one-dimensional subgroup of G corresponding to β. Given $\mu \in \chi_*(T) \otimes \mathbb{Q} \equiv \chi(T) \otimes \mathbb{Q}$, we denote by $P(\mu)$ the parabolic subgroup of G generated by T and by the subgroups U_β corresponding to the roots β such that $(\beta, \mu) \geq 0$. Given a parabolic subgroup $P = P(\mu)$, sometimes we denote by P^- the opposite standard parabolic subgroup, namely $P(-\mu)$.

1.2 Colored fans of smooth complete symmetric varieties with Picard number one

In [Ru2] we have proved the following combinatorial classification of the smooth complete symmetric varieties with Picard number one. *Unless otherwise specified, we always suppose G simply connected.* Notice that this assumption is not restrictive (see, for example, [V1], §2.1).

Theorem 1.1 *Let G be a semisimple, simply connected group and let G/H be a homogeneous symmetric variety. Suppose that there is a smooth, complete embedding X of G/H with Picard number one. Then:*

- *Fixed G/H, there is, up to equivariant isomorphism, at most one embedding with the previous properties. Moreover, it is projective and contains at most two closed orbits.*
- *The number of colors of G/H is equal to the rank l of G/H; in particular there are no exceptional roots.*
- *We have the following classification depending on the type of the restricted root system $R_{G,\theta}$:*
  - *(i) If $R_{G,\theta}$ has type $A_1 \times A_1$, then $\chi(S)$ has basis $\{2\omega_1, \omega_1+\omega_2\}$; in particular H has index two in $N_G(G^\theta)$. Moreover, X has two closed orbits; the maximal colored cones of the colored fan of X are $(cone(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_1^\vee}\})$ and $(cone(\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_2^\vee}\})$.*
  - *(ii) If $l = 1$, then G/H can be isomorphic neither to $SL_{n+1}/S(L_1 \times L_n)$, nor to SL_2/SO_2. With such hypothesis, G/H has a unique non trivial embedding which is simple, projective, smooth and with Picard number one.*
  - *(iii) If $R_{G,\theta}$ has type A_l with $l > 1$, we have the following possibilities:*
    - *(a) $H = N_G(G^\theta)$ and X is simple. In this case X is associated either to the colored cone $(cone(\alpha_1^\vee, ..., \alpha_{l-1}^\vee, -\omega_1^\vee), \{D_{\alpha_1^\vee}, ..., D_{\alpha_{l-1}^\vee}\})$ or to the one $(cone(\alpha_2^\vee, ..., \alpha_l^\vee, -\omega_l^\vee), \{D_{\alpha_2^\vee}, ..., D_{\alpha_l^\vee}\})$;*

(b) $H = G^\theta$ *and* $l = 2$. *In this case* X *has two closed orbits. The maximal colored cones of the colored fan of* X *are* $(cone(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_1^\vee}\})$ *and* $(cone(\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_2^\vee}\})$.

(iv) *If* $R_{G,\theta}$ *has type* B_2*, then* X *is simple and we have the following possibilities:*

(a) $H = N_G(G^\theta)$ *and* X *is associated to* $(cone(\alpha_1^\vee, -\omega_1^\vee),\ \{D_{\alpha_1^\vee}\})$;

(b) $H = G^\theta$ *and* X *is associated to* $(cone(\alpha_2^\vee, -\omega_2^\vee),\ \{D_{\alpha_2^\vee}\})$. *Moreover* G/H *cannot be Hermitian.*

(v) *If* $R_{G,\theta}$ *has type* B_l *with* $l > 2$*, then* $H = N_G(G^\theta)$*,* X *is simple and is associated to* $(cone(\alpha_1^\vee, ..., \alpha_{l-1}^\vee, -\omega_1^\vee), \{D_{\alpha_1^\vee}, ..., D_{\alpha_{l-1}^\vee}\})$.

(vi) *If* $R_{G,\theta}$ *has type* C_l*, then* $H = G^\theta$*,* X *is simple and corresponds to* $(cone(\alpha_1^\vee, ..., \alpha_{l-1}^\vee, -\omega_1^\vee), \{D_{\alpha_1^\vee}, ..., D_{\alpha_{l-1}^\vee}\})$. *Moreover,* G/H *cannot be Hermitian.*

(vii) *If* $R_{G,\theta}$ *has type* BC_l *with* $l > 1$*, then* $H = N_G(G^\theta) = G^\theta$*,* X *is simple and corresponds to* $(cone(\alpha_1^\vee, ..., \alpha_{l-1}^\vee, -\omega_1^\vee),\ \{D_{\alpha_1^\vee}, ..., D_{\alpha_{l-1}^\vee}\})$.

(viii) *If* $R_{G,\theta}$ *has type* D_l *with* $l > 4$*, then* $\chi_*(S)$ *is freely generated by* $\omega_1^\vee, ..., \omega_{l-2}^\vee, \omega_{l-1}^\vee + \omega_l^\vee, 2\omega_l^\vee$*; in particular* H *has index two in* $N_G(G^\theta)$. X *has two closed orbits; the maximal colored cones of the colored fan of* X *are* $(cone(\alpha_1^\vee, ..., \alpha_{l-1}^\vee, -\omega_1^\vee),\ \{D_{\alpha_1^\vee}, ..., D_{\alpha_{l-1}^\vee}\})$ *and* $(cone(\alpha_1^\vee, ..., \alpha_{l-2}^\vee,\ \alpha_l^\vee, -\omega_1^\vee),\ \{D_{\alpha_1^\vee}, ..., D_{\alpha_{l-2}^\vee},\ D_{\alpha_l^\vee}\})$.

(ix) *If* $R_{G,\theta}$ *has type* D_4*, then* H *has index two in* $N_G(G^\theta)$ *and* X *has two closed orbits. If* $\chi_*(S) = \mathbb{Z}\omega_i^\vee \oplus \mathbb{Z}\omega_2^\vee \oplus \mathbb{Z}(\omega_j^\vee + \omega_k^\vee) \oplus \mathbb{Z}2\omega_k^\vee$*, then the maximal colored cones of the colored fan of* X *are* $(cone(\alpha_i^\vee, \alpha_2^\vee, \alpha_j^\vee, -\omega_i^\vee),\ \{D_{\alpha_i^\vee}, D_{\alpha_2^\vee}, D_{\alpha_j^\vee}\})$ *and* $(cone(\alpha_i^\vee, \alpha_2^\vee, \alpha_k^\vee, -\omega_i^\vee),\ \{D_{\alpha_i^\vee}, D_{\alpha_2^\vee}, D_{\alpha_k^\vee}\})$.

(x) *If* $R_{G,\theta}$ *has type* G_2 *then* $H = N_G(G^\theta) = G^\theta$*,* X *is simple and is associated to* $(cone(\alpha_2^\vee, -\omega_2^\vee), \{D_{\alpha_2^\vee}\})$.

(xi) *If the type of* $R_{G,\theta}$ *is different from the previous ones, then there is no a variety* X *with the requested properties.*

1.3 Description of some exceptional groups via composition algebras

We will need a description of some exceptional groups via complex composition algebras and Jordan algebras. The interested reader can see [LM] and [Ad] for a detailed exposition of the facts which we recall here.

Let $\mathbb{A}$ be a complex composition algebra, i.e. $\mathbb{A} = \mathbb{A}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ where $\mathbb{A}_\mathbb{R}$ is a real division algebra (namely $\mathbb{A}_\mathbb{R} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$). If $a \in \mathbb{A}$, let $\bar{a}$ be its conjugate; we denote by $Im\mathbb{A}$ the subspace of pure imaginary element, i.e. the elements a such that $\bar{a} = -a$. Let $\mathcal{J}_3(\mathbb{A})$ be the space of $\mathbb{A}$-Hermitian matrices of order three, with coefficients in $\mathbb{A}$:

$$\mathcal{J}_3(\mathbb{A}) = \left\{ \begin{pmatrix} r_1 & \bar{x}_3 & \bar{x}_2 \\ x_3 & r_2 & \bar{x}_1 \\ x_2 & x_1 & r_3 \end{pmatrix}, r_i \in \mathbb{C}, x_i \in \mathbb{A} \right\}.$$

$\mathcal{J}_3(\mathbb{A})$ has the structure of a Jordan algebra with the multiplication $A \circ B := \frac{1}{2}(AB + BA)$, where AB is the usual matrix multiplication. There is a well defined cubic form on $\mathcal{J}_3(\mathbb{A})$, which we call the determinant. Given $P \in \mathcal{J}_3(\mathbb{A})$, its comatrix is defined by

$$com\, P = P^2 - (trace\, P)P + \frac{1}{2}((trace\, P)^2 - trace\, P^2)I$$

and characterized by the identity $com(P)P = det(P)I$. In particular, P is invertible if and only if $det(P)$ is different from 0.

Let $SL_3(\mathbb{A}) \subset GL_{\mathbb{C}}(\mathcal{J}_3(\mathbb{A}))$ be subgroup preserving the determinant; $\mathcal{J}_3(\mathbb{A})$ is an irreducible $SL_3(\mathbb{A})$ representation. We let $SO_3(\mathbb{A})$ denote the group of complex linear transformations preserving the Jordan multiplication; it is also the subgroup of $SL_3(\mathbb{A})$ preserving the quadratic form $Q(A) = trace(A^2)$.

We call $\mathcal{Z}_2(\mathbb{A}) := \mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})^* \oplus \mathbb{C}^*$ the space of Zorn matrices. The space $\mathfrak{sp}_6(\mathbb{A}) := \mathbb{C}^* \oplus \mathcal{J}_3(\mathbb{A})^* \oplus (Lie(SL_3(\mathbb{A})) + \mathbb{C}) \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathbb{C}$ has a structure of Lie algebra and $\mathcal{Z}_2$ has a natural structure of (simple) $\mathfrak{sp}_6(\mathbb{A})$-module. There is a unique closed connected subgroup of $GL_{\mathbb{A}}(\mathcal{Z}_2(\mathbb{A}))$ with Lie algebra $\mathfrak{sp}_6(\mathbb{A})$; we denote it by $Sp_6(\mathbb{A})$. Moreover, there is a $Sp_6(\mathbb{A})$-invariant symplectic form on $\mathcal{Z}_2(\mathbb{A})$.

The closed $Sp_6(\mathbb{A})$-orbit in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$ is the image of the $Sp_6(\mathbb{A})$-equivariant rational map:

$$\begin{array}{cccc} \phi : \mathbb{P}(\mathbb{C} \oplus \mathcal{J}_3(\mathbb{A})) & \dashrightarrow & \mathbb{P}(\mathcal{Z}_2(\mathbb{A})) \\ (x, P) & \rightarrow & (x^3, x^2 P, x\, com(P), det(P)). \end{array}$$

Furthermore, if $\mathbb{C}$ is interpreted as a space of diagonal matrix (in $\mathcal{J}_3(\mathbb{A})$) and (I, P) is interpreted as a matrix of three row vectors in $\mathbb{A}^6$, then the previous map is the usual Plucker map. The condition $P \in \mathcal{J}_3(\mathbb{A})$ can be interpreted as the fact that the three vectors defined by the matrix (I, P) are orthogonal with respect to the Hermitian symplectic two-form $w(x, y) =^t xJ\bar{y}$, where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Therefore, it is natural to see the closed $Sp_6(\mathbb{A})$-orbit in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$ as the Grassmannian $\mathbb{LG}(\mathbb{A}^3, \mathbb{A}^6)$ of symplectic 3-planes in $\mathbb{A}^6$.

Explicitly, we have the following possibilities: 1) if $\mathbb{A}_R = \mathbb{R}$ then $SL_3(\mathbb{A})$ is SL_3, $SO_3(\mathbb{A})$ is SO_3 and $Sp_6(\mathbb{A})$ is Sp_6; 2) if $\mathbb{A}_R = \mathbb{C}$ then $SL_3(\mathbb{A})$ is $SL_3 \times SL_3$, $SO_3(\mathbb{A})$ is SL_3 and $Sp_6(\mathbb{A})$ is SL_6; 3) if $\mathbb{A}_R = \mathbb{H}$ then $SL_3(\mathbb{A})$ is SL_6, $SO_3(\mathbb{A})$ is Sp_6 and $Sp_6(\mathbb{A})$ is $Spin_{12}$; 4) if $\mathbb{A}_R = \mathbb{O}$ then $SL_3(\mathbb{A})$ is E_6, $SO_3(\mathbb{A})$ is F_4 and $Sp_6(\mathbb{A})$ is E_7.

1.4 Generalized flag varieties

We conclude this section with a description of the projective homogeneous varieties with Picard number one for the classic groups. Let V be a n-dimensional

vector space, we will denote by $\mathbb{G}_m(V)$ the *Grassmannian of the m-dimensional subspace* of V. Let q be a non-degenerate symmetric bilinear form on V and let $SO(V,q)$ be the corresponding special orthogonal group. We will say that a subspace of V is isotropic if the restriction of q to it is zero. Fix an integer m such that $2m \leq r$ and let $\mathbb{IG}_m(V) \subset \mathbb{G}_m(V)$ be the algebraic subvariety whose points are identified with isotropic m-dimensional subspaces of V. The group $SO(V,q)$ acts on $\mathbb{IG}_m(V)$ in the natural manner and the action is transitive if $2m < n$. Instead, if $2m = n$, $\mathbb{IG}_m(V)$ consists of two isomorphic $SO(V,q)$-orbits (each of them being a connected component of $\mathbb{IG}_m(V)$); we denote a such orbit by $\mathbb{S}_m(V)$ (or by $\mathbb{S}_m$ if $V = \mathbb{C}^{2m}$ and q is the standard symmetric bilinear form). The variety $\mathbb{IG}_m(V)$ is called the *isotropic Grassmannian of m-dimensional isotropic subspace*, while $\mathbb{S}_m(V)$ is called the *spinorial variety of order m*. In an analogous manner, let q' be a non-degenerate skew-symmetric bilinear form on V and fix an integer m such that $2m \leq n$. We denote by $\mathbb{LG}_m(V)$ the algebraic subvariety of $\mathbb{G}_m(V)$ whose points are identified with isotropic m-dimensional subspaces of V; $\mathbb{LG}_m(V)$ is called the *Lagrangian Grassmannian*.

2 Varieties of rank one

In this section we describe the varieties with rank one. For any homogenous symmetric variety G/H with rank one there is a unique equivariant completion X. It is smooth, projective and has Picard number at most two. Moreover, X has exactly two orbits: an open orbit and a closed orbit of codimension one. Thus X is a homogeneous variety by the following theorem due to D. Ahiezer.

Theorem 2.1 (see [A1], Theorem 4) *Let G be a semisimple group and let H be a closed reductive subgroup. Let X be an equivariant (normal) completion of G/H such that $X \setminus (G/H)$ is a G-orbit (of codimension one). Then X is a homogeneous space for a larger group. If G/H is a symmetric variety and G acts effectively on G/H, we have the following possibilities:*

1. *G is $SL_2 \times SL_2$, H is SL_2 and X is $\{(x,t)\,|\,det\,(x) = t^2\} \subset \mathbb{P}(S^2\mathbb{C}^2 \oplus \mathbb{C})$;*
2. *G is $PSL_2 \times PSL_2$, H is PSL_2 and X is $\mathbb{P}(S^2\mathbb{C}^2)$;*
3. *G is SL_n, H is GL_{n-1}, θ has type AIV (or AI if $n = 2$) and X is $\mathbb{P}(\mathbb{C}^n) \times \mathbb{P}((\mathbb{C}^n)^*)$;*
4. *G is PSL_2, H is PSO_2, θ has type AI and X is $\mathbb{P}(\mathfrak{sl}_2)$;*
5. *G is Sp_{2n}, H is $Sp_2 \times Sp_{2n-2}$, θ has type CII and X is $\mathbb{G}_2(2n)$;*
6. *G is SO_n, H is SO_{n-1}, θ has type BII or DII and X is $\{(x,t)\,|\,q(x,x) = t^2\} \subset \mathbb{P}(\mathbb{C}^n \oplus \mathbb{C})$, where q is the standard symmetric bilinear form;*
7. *G is SO_n, H is $S(O_1 \times O_{n-1})$, θ has type BII or DII, and X is $\mathbb{P}(\mathbb{C}^n)$;*
8. *G is F_4, H is $Spin_9$, θ has type FII and X is $E_6/P_{\omega_1} \equiv \mathbb{P}^2\mathbb{O}$;*

3 Varieties of rank two

In the following of this work we always suppose that G/H has rank strictly greater than one. In this section we describe the varieties of rank two which do not belong to an infinite family. Explicitly, we consider completion of the following homogenous varieties: 1) the symmetric variety $SL_4/N_{SL_4}(SL_2 \times SL_2)$ of type $AIII$; 2) the symmetric $(SL_3 \times SL_3)$-variety SL_3; 3) the symmetric variety SL_3/SO_3 of type AI; 4) the symmetric variety SL_4/Sp_4 of type AII; 5) the symmetric variety E_6/F_4 of type EIV (with E_6 simply connected); 6) the symmetric variety $E_6/N_{E_6}(F_4)$ of type EIV; 7) the symmetric variety $Sp_4/(Sp_2 \times Sp_2)$ of type CII; 8) the symmetric variety G_2; 9) the symmetric variety $G_2/(Sl_2 \times Sl_2)$ of type G; 10) the symmetric varieties whose restricted root system has type $A_1 \times A_1$. In the last case $G/N_G(G^\theta)$ is isomorphic to $SO_n/N_{SO_n}(SO_{n-1}) \times SO_m/N_{SO_m}(SO_{m-1})$; n, m are strictly greater than two and H has index two in $N_G(G^\theta)$.

In this section we do not consider the completions of the symmetric varieties $Spin_5$ and $SO_8/N_{SO_8}(GL_4)$ because: 1) $Spin_5$ is isomorphic to Sp_4 and we will study it together with the completion of Sp_{2n}; 2) $SO_8/\ N_{SO_8}(GL_4)$ is isomorphic to $SO_8/N_{SO_8}(SO_2 \times SO_6)$ and we will study it together with the completion of $SO_n/N_{SO_n}(SO_2 \times SO_{n-2})$.

3.1 Some general considerations

We begin with some general considerations; in particular we do not suppose yet that X has rank two. We prove that a smooth projective symmetric variety X with Picard number one is a homogeneous variety if and only if $H^0(G/P^-, N_{G/P^-,X}) \neq 0$ for an appropriate closed orbit G/P^- of X. Furthermore, we explain how to study the automorphism group of X (if such variety is not homogenous).

Lemma 3.1 *Let X be a smooth projective symmetric variety with Picard number one and let Y be a proper G-stable closed subvariety of X. If Y is smooth, then it is a G-orbit.*

Proof. Recall that the minimal B-stable affine open set X_B which intersects Y is a product $P_u \times Z$, where Z is an affine L-symmetric variety. One can easily show that Z is an irreducible L-representation (see [Ru2], pages 9 and 17 or [Ru2], Theorem 2.2). Observe that Y is smooth if and only if $Y \cap Z$ is smooth. But $Y \cap Z$ is a cone because the center of L acts non-trivially on it. Thus $Y \cap Z$ is smooth if and only if it is a subrepresentation of Z. Since Z is irreducible, $Y \cap Z$ is smooth if and only if it is a point. In such case Y is a closed orbit. □

We use the previous lemma only to prove following corollary.

Corollary 3.1 *Let X be a smooth projective symmetric variety with Picard number one. If $Aut^0(X)$ does not act transitively over X, then $Aut^0(X)$ stabilizes a closed G-orbit.*

Proof. Let Y be a minimal closed G-stable subvariety stabilized by $Aut^0(X)$. Suppose by contradiction that Y is not a G-orbit, then it is singular by the Lemma 3.1. Thus the singular locus of Y is not empty and is stabilized by $Aut^0(X)$. In particular, its irreducible components are $Aut^0(X)$-stable, closed subvarieties of X, properly contained in Y; a contradiction. □

A closed orbit G/P^- of X is stabilized by $Aut^0(X)$ if and only if $H^0(G/P^-, N_{G/P^-,X}) = 0$, where $N_{G/P^-,X}$ is the normal bundle to G/P^- (see [A2], §2.3). Recall that X_B is $P_u \times Z$; moreover, the intersection of G/P^- with X_B is $P_u \times \{0\}$. Thus we can identify Z with the fibre of the normal bundle $N_{G/P^-,X}$ over P^-/P^-. We know by the Borel-Weil theorem (see [A2], §4.3) that $H^0(G/P^-, N_{G/P^-,X}) = 0$ if and only if the highest weight of Z is not dominant.

Now, we want to explain how to calculate the highest weight ω of Z when the rank of G/H is two. The center of L has dimension one and $R_{[L,L],\theta}$ has rank one; moreover, if α is the simple restricted root of $R_{[L,L],\theta}$, then $(\omega, \alpha^\vee) = 1$ (see [Ru2], Theorem 2.2). Let $\varpi^\vee$ be a generator of $\chi_*(Z(L)^0/(Z(L)^0 \cap H))$; suppose also that there exists $x' := lim_{t\to 0}\varpi^\vee(t) \cdot x_0$ in the closure of $T \cdot x_0$ in Z. Then $(\omega, \varpi^\vee) = 1$ because $(Z(L)^0/(Z(L)^0 \cap H)) \cup x'$ is contained in Z (and Z has a unique isotopic component as $Z(L)^0$-representation). To determine the sign of $\varpi^\vee$, we will use the fact that $(\lambda, \varpi^\vee) \geq 0$ for every $\lambda \in \chi(S) \cap \sigma^\vee$, where σ is the cone associated to the closure of $T \cdot x_0$ in Z and $\sigma^\vee$ is its dual cone. In [Ru2] is proved that $\sigma^\vee$ is equal to $W_{L,\theta} \cdot C(X)^\vee$, where $W_{L,\theta}$ is the Weyl group of the restricted root system of L (see the proof of Lemma 2.8 in [Ru2]).

In the following of this section, let X be a smooth projective symmetric variety with Picard number one, over which $Aut^0(X)$ acts non-transitively. We will prove in §3 that X has rank 2; for the moment let us assume it. Suppose also that $Aut^0(X)$ stabilizes all the closed G-orbits in X. We define the canonical completion of G/H as the simple symmetric variety associated to the cone $(-C^+, \emptyset)$. Let $\widetilde{X}$ be the decoloration of X, namely the minimal toroidal variety with a proper map $\pi : \widetilde{X} \to X$ which extents the identity over G/H: it is the canonical variety if X is simple and the blow-up of the canonical variety in the closed orbit otherwise. Moreover, $\widetilde{X}$ is the blow-up of X along the closed orbits (see Theorem 3.3 in [Br2]); in particular, $\widetilde{X}$ is smooth and the G-stable divisor of X is the image of a G-stable divisor of $\widetilde{X}$. Let $G/\widetilde{P}^-$ be a closed orbit of $\widetilde{X}$.

We claim that, if $Aut^0(G/\widetilde{P}^-) = G/Z(G)$, then $Lie(Aut^0(X))$ is isomorphic to $Lie(G)$. The group $Aut^0(X)$ is isomorphic to $Aut^0(\widetilde{X})$. Indeed $Aut^0(X)$ is contained in $Aut^0(\widetilde{X})$ because the closed orbits are stable under the action of $Aut^0(X)$. Moreover, by a result of A. Blanchard, $Aut^0(\widetilde{X})$ acts on X in such a way that the projection is equivariant (see [A2], §2.4). Since $\widetilde{X}$ is complete and $Pic\,(\widetilde{X})$ is discrete, the group $Aut^0(\widetilde{X})$ is linear algebraic and its Lie algebra is the space of global vector fields, namely $H^0(X, \mathcal{T}_{\widetilde{X}})$.

We want to prove that $Aut^0(\widetilde{X})$ stabilizes all the G-orbits in $\widetilde{X}$. This fact implies that $Aut^0(\widetilde{X})$ is reductive by a result of M. Brion (see [Br4], Theorem 4.4.1). First, suppose X simple; we will prove in §3.2 that H is $N_G(G^\theta)$. Thus

$\widetilde{X}$ is a wonderful G-variety. Moreover, $Aut^0(\widetilde{X})$ is semisimple, $\widetilde{X}$ is a wonderful $Aut^0(X)$-variety and the set of colors is $D(G/H)$ (see [Br4], Theorem 2.4.2). In [Br4] it is also determined the automorphism group of the wonderful completion of a simple adjoint group $\overline{G}$ (see [Br4], Example 2.4.5); in particular, it is proved that $Aut^0(\overline{G})$ is $\overline{G} \times \overline{G}$ if $\overline{G} \neq PSL(2)$. Coming back to our problem, the wonderful G-variety $\widetilde{X}$ has two stable prime divisors: the exceptional divisor E and the strict transform $\widetilde{D}$ of the G-stable divisor D of X. The divisor $\widetilde{D}$ corresponds to a simple restricted root (see [dCP1], Lemma 2.2), which is not a dominant weight because the restricted root system is irreducible of rank 2 (see Theorem 1.1). Thus $\widetilde{D}$ is fixed by $Aut^0(\widetilde{X})$ (see [Br4], Theorem 2.4.1), so $Aut^0(\widetilde{X})$ fixes all the G-orbits in $\widetilde{X}$.

We can proceed similarly in the case where X is non-simple. Indeed, using [Br1], Proposition 3.3, [V1], §2.4 Proposition 1 and Proposition 2, one can easily prove that $H^0(\widetilde{X}, \mathcal{O}(F)) = \mathbb{C}s_F$ for each G-stable divisor F of $\widetilde{X}$ (here s_F is a global section with divisor F). Since $\xi \cdot s_F$ is a scalar multiple of s_F for any $\xi \in H^0(\widetilde{X}, \mathcal{T}_{\widetilde{X}}) = Lie(Aut^\circ(\widetilde{X}))$, F is stabilized by $Aut^\circ(\widetilde{X})$ as in the proof of Theorem 2.4.1 in [Br4] (see also Proposition 4.1.1 in [BB]).

Notice that an element of $Z(G)$ acts trivially on G/H (and on $\widetilde{X}$) if and only if it belongs to H, thus $Aut^\circ(\widetilde{X})$ contains $G/(Z(G) \cap H)$. Assume that $Aut^0(G/\widetilde{P}^-)$ is $G/Z(G)$, then the restriction $\psi : Aut^\circ(\widetilde{X}) \to Aut^\circ(G/\widetilde{P}^-) = G/Z(G)$ is an isogeny, because $Aut^0(\widetilde{X})$ stabilizes all the G-orbits. Indeed, $ker\, \psi$ centralizes $G/(Z(G) \cap H)$ (because $Aut^0(\widetilde{X})$ is reductive) and stabilizes G/H, so it is contained in the finite group $Aut_G(G/H) \cong N_G(H)/H$. Therefore $Aut^\circ(X)$ is $G/(Z(G) \cap H)$.

Remark 3.1.1 To summarize, given X over which $Aut^0(X)$ acts non-transitively, we have to prove:

- $Aut^0(X)$ stabilizes all the closed G-orbits in X;
- X has rank 2;
- $Aut(G/\widetilde{P}^-) = G/Z(G)$ for a closed orbit $G/\widetilde{P}^-$ of $\widetilde{X}$.

If the previous conditions are verified, then $Aut^0(X)$ is $G/(Z(G) \cap H)$.

In the following of this section we suppose that the previous conditions are verified and we try to study the full automorphism group $Aut(X)$. This group permutes the $Aut^0(X)$-orbits; in particular it stabilizes the open orbit and the codimension one orbit. Furthermore, $Aut(X)$ stabilizes G/H, thus it is contained in $Aut(G/H)$. We will prove the converse. Notice that $Aut(X)$ is contained in $Aut(\widetilde{X})$, because $\widetilde{X}$ is the blow-up of X along a $Aut(X)$-stable (eventually non-connected) subvariety.

Remark 3.1.2 *The automorphism group of $\widetilde{X}$ is isomorphic to the automorphism group of G/H.* Indeed, $Aut(\widetilde{X})$ stabilizes G/H, so it is contained in $Aut(G/H)$. Moreover, we can extend every automorphism of G/H to an automorphism of $\widetilde{X}$, because $\widetilde{X}$ is normal and toroidal (these facts imply that each

(non-open) G-orbit $\mathcal{O}$ is contained in the closure of an orbit of dimension equal to $dim\,\mathcal{O}+1$). To study $Aut(X)$, we have to determine the automorphisms of $\widetilde{X}$ which descend to X.

Now, we study $Aut(\widetilde{X})$ (and $Aut(G/H)$). For any φ in $Aut(\widetilde{X})$ we define $\tilde{\varphi} \in Aut_{alg}(Aut^{\circ}(\widetilde{X})) \equiv Aut_{alg}(G)$ by

$$\tilde{\varphi}(g) \cdot x = \varphi(g \cdot \varphi^{-1}(x)), \quad g \in Aut^0(X),\ x \in \widetilde{X}.$$

The kernel of $\varphi \to \tilde{\varphi}$ is the group of equivariant automorphism of $\widetilde{X}$ (and G/H), thus it is isomorphic to $N_G(H)/H$. We will prove that $N_G(H)/H$ is simple (or trivial); moreover $Z(G/Z(G)\cap H)$ is trivial only if $N_G(H)/H$ is it. Thus $ker(\varphi \to \tilde{\varphi})$ is $Z(G/Z(G)\cap H) = Z(G)/(Z(G)\cap H)$.

Let T'' be a maximal torus of H, then its centralizer $T' := C_G(T'')$ is a maximal torus of G (see [T2], Lemma 26.2); moreover T'' contains a regular one-parameter subgroup λ of G. Thus $B' := P(\lambda)$ is a Borel subgroup of G and $B'' := (B' \cap H)^0$ is a Borel subgroup of H. The group $Aut_{alg}(G)$ is generated by $\overline{G} := G/Z(G)$ and $E = \{\psi \in Aut_{alg}(G) : \psi(T') = T' \text{ and } \psi(B') = B'\}$; the intersection of E with $\overline{G}$ is $\overline{T}\,' := T'/Z(G)$, so $Aut_{alg}(G)$ is the semidirect product of $\overline{G}$ and $E' := \{\psi \in E : \psi(t) = t\ \forall t \in T'\}$. Observe that every $\psi \in E$ induces an automorphism of the Dynkin diagram (with respect to T' and B'), moreover such automorphism is trivial if and only if ψ belongs to $\overline{G}$. Furthermore, $Aut(\widetilde{X})$ is generated by $G/(Z(G)\cap H)$ and by the stabilizer of x_0 in $Aut(\widetilde{X})$. Notice that, given $\varphi \in Aut_{x_0}(\widetilde{X})$, then $\widetilde{\varphi}$ belongs to E, up to composing φ with an element of H.

Remark 3.1.3 Suppose that $Aut_G(\widetilde{X})$ is $Z(G/(Z(G)\cap H))$; in particular it is contained in $Aut^0(\widetilde{X})$. Then $Aut(G/H)$ is generated by $Aut^0(G/H) = G/(Z(G)\cap H)$ and by the subgroup $K = \{\varphi \in Aut(G/H) : \varphi(x_0) = x_0 \text{ and } \widetilde{\varphi} \in E\}$. Moreover the map $\varphi \to \tilde{\varphi}$ restricted to K is injective, because $Z(Aut(G/H))$ does not fix x_0. Observe that any automorphism of G stabilizing H induces an automorphism of G/H, so K is isomorphic to $K' := \{\psi \in E :\ \psi(H) = H\}$. Besides θ belongs to K (see [T2], Lemma 26.2).

Now, we explain how to prove that every automorphism of $\widetilde{X}$ descends to an automorphism of X. Let φ in K, then $\varphi(g\widetilde{P}') = \widetilde{\varphi}(g)\widetilde{P}'$ for every $g\widetilde{P}' \in G/\widetilde{P}'$. First, suppose $\widetilde{X}$ (and X) simple; in particular, φ stabilizes the closed orbit of $\widetilde{X}$. Let x_1 be a point (in the closed orbit of $\widetilde{X}$) fixed by B' and let $\widetilde{P}'$ be the stabilizer of x_1 in G. Then φ fixes x_1, because $\widetilde{\varphi} \in E$. Let G/P' be the image of $G/\widetilde{P}' \subset \widetilde{X}$ in X; we have to prove that $\widetilde{\varphi}(P')$ is P'. Thus, it is sufficient that φ induces the trivial automorphism of the Dynkin diagram (with respect to T' and B').

Finally, suppose that X contains two closed orbits. In this case the restricted root system has type A_2, because of Theorem 1.1 and Proposition 3.4. Furthermore, the closed orbits are $G/P(\omega_1)$ and $G/P(\omega_2)$; in particular, they are not G-isomorphic. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the inverse images in $\widetilde{X}$ respectively of G/P_1 and G/P_2; one can easily show that they are isomorphic. We want to prove that

any automorphism in K exchange $\mathcal{O}_1$ with $\mathcal{O}_2$. Let $\widetilde{D}$ be the strict trasform of D and let E_1, E_2 be the exceptional divisors of $\widetilde{X}$; we can suppose that $\mathcal{O}_i$ is the intersection of $\widetilde{D}$ with E_i. Thus it is sufficient to prove that φ exchange E_1 with E_2. For the moment, assume this fact and let x_i be a point in $\mathcal{O}_i$ stabilized by B', let P'_i be the stabilizer of $\pi(x_i)$ and let P' be the stabilizer of x_1 (and x_2). We have $\varphi(x_1) = x_2$, because $\widetilde{\varphi}$ belongs to E. Now, it is sufficient to prove that $\widetilde{\varphi}$ is associated to a non-trivial automorphism of the Dynkin diagram. Indeed, in this case $\widetilde{\varphi}(P'_1)$ is a parabolic subgroup of G containing B', distinct by P'_1 and with the same dimension of P'_1; thus it is P'_2.

Remark 3.1.4 Summarizing, to prove that $Aut(X)$ coincides with $Aut(G/H)$ (and with $Aut(\widetilde{X})$), it is sufficient to show that:

- $N(H)/H$ is simple or trivial;
- if $N(H)/H$ is non-trivial, then also $Z(G)/(Z(G) \cap H)$ is non-trivial;
- if X is simple (and non-homogeneous), then $R_{G,\theta}$ has type G_2. Moreover, given $\varphi \in K$ then $\widetilde{\varphi}$ induces a trivial automorphism of the Dynkin diagram (associated to T' and B');
- if X is non-simple and $\varphi \in K$, then φ exchange E_1 with E_2. Moreover, $\widetilde{\varphi}$ induces a non-trivial automorphism of the Dynkin diagram (associated to T' and B').

To study $Aut(X)$, we have to determine the group $K \subset Aut(G/H)$. We will prove that θ is always contained $Aut(X)$. If, moreover, G/H is not isomorphic to SL_3, then $K' \cap E'$ is E'; thus the map $\varphi \to \widetilde{\varphi}$ is surjective (if $G/H \neq SL_3$). In following subsections we will study $Aut(X)$ by a case-to-case analysis.

3.2 Homogeneous varieties

We begin studying the symmetric varieties of rank 2 which are homogeneous.

Proposition 3.1 *The smooth completion of $SL_4/N_{SL_4}(S(L_2 \times L_2))$ with Picard number one is isomorphic to $\mathbb{G}_2(6)$.*

Proof. The symmetric varieties $SL_4/N_{SL_4}(S(L_2 \times L_2))$ has type $AIII$. The group SL_4 acts on the six-dimensional space $\bigwedge^2(\mathbb{C}^4)$, thus it acts on $\mathbb{G}_2(\bigwedge^2 \mathbb{C}^4)$. The stabilizer of the space generated by $e_1 \wedge e_2$ and $e_2 \wedge e_3$ is $N_{SL_4}(S(L_2 \times L_2))$, thus $SL_4/N_{SL_4}(S(L_2 \times L_2))$ is contained in $\mathbb{G}_2(\bigwedge^2 \mathbb{C}^4)$. Moreover $SL_4/N_{SL_4}(S(L_2 \times L_2))$ has the same dimension of $\mathbb{G}_2(\bigwedge^2 \mathbb{C}^4)$, so the Grasmannian is the unique smooth completion of $SL_4/N_{SL_4}(S(L_2 \times L_2))$ with Picard number one. □

Proposition 3.2 *The smooth completion of $E_6/N_{E_6}(F_4)$ with Picard number one is isomorphic to $\mathbb{P}(\mathbb{J}_3(\mathbb{O}))$.*

Proof. The symmetric varieties $E_6/N_{E_6}(F_4)$ has type EIV and $\mathbb{J}_3(\mathbb{O})$ is the 27-dimensional irreducible representation of E_6 corresponding to the first fundamental weight. The subgroup F_4 of E_6 is isomorphic to the group of automorphism of $\mathbb{J}_3(\mathbb{O})$; in particular F_4 fixes the identity matrix. Thus $\mathbb{P}(\mathbb{J}_3(\mathbb{O}))$ contains the 26-dimensional variety G/H. $\square$

Proposition 3.3 *The smooth completion of $Sp_8/N_{Sp_8}(Sp_4 \times Sp_4)$ with Picard number one is isomorphic to $E_6/P_1 \equiv \mathbb{P}^2(\mathbb{O})$.*

Proof. The symmetric variety $Sp_8/N_{Sp_8}(Sp_4 \times Sp_4)$ has type CII. Let (G', σ) be an involution of type EI, where G' is the simply connected simple group of type E_6 and G'^{σ} has type C_4. Choose a maximally σ-split torus and a Borel subgroup of G' as in §1.1.1; then σ acts as $-id$ over $R_{G'}$ and the parabolic subgroup $P := P(\varpi_1)$ of G' is opposed to $\sigma(P)$. Furthermore, $P \cap \sigma(P)$ is a Levi subgroup of P containing P^{σ}; the derived subgroup of $P \cap \sigma(P)$ has type D_5 while $(P^{\sigma})^0$ has type $B_2 \times B_2$. Hence G'^{σ}/P^{σ} is isomorphic to $Sp_8/N_{Sp_8}(Sp_4 \times Sp_4)$, up to quotient by a finite group. Notice that G'/P is a smooth completion of G'^{σ}/P^{σ} with Picard number one. The variety G'^{σ}/P^{σ} cannot be isomorphic to $Sp_8/(Sp_4 \times Sp_4)$ because the unique smooth completion of $Sp_8/(Sp_4 \times Sp_4)$ with Picard number one is isomorphic to $\mathbb{G}_4(8)$ (see Proposition 4.5 in § 4.3 and Theorem 1.1). $\square$

Now, we study the symmetric varieties whose restricted root system is reducible. If the restricted root system of G/H has type A_1, then $G/N_G(G^{\theta})$ is isomorphic to $SO_n/N_{SO_n}(SO_{n-1})$ for an appropriate $n \geq 3$. If $n = 3$, the type of (G, θ) is AI; if $n = 4$ then $G/N_G(G^{\theta})$ is isomorphic to PSL_2; if $n = 6$, the type of (G, θ) is AII; if $n = 2k$ with $k \geq 4$, the type of (G, θ) is DII; if $n = 2k + 1$ with $k \geq 2$, the type of (G, θ) is BII.

Proposition 3.4 *If $R_{G,\theta}$ has type $A_1 \times A_1$, then $G/N_G(G^{\theta})$ is isomorphic to $SO_n/N_{SO_n}(SO_{n-1}) \times SO_m/N_{SO_m}(SO_{m-1})$ and the smooth projective embedding of G/H with Picard number one is isomorphic to $\mathbb{IG}_1(n+m)$.*

First we want to describe H: write $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ and let $g_i \in N_{G_i}(G_i^{\theta})$ be a representant of the non-trivial element of $N_{G_i}(G_i^{\theta})/G_i^{\theta}$ for each i. Then H is generated by G^{θ} and (g_1, g_2).

Proof. The group $SO_n \times SO_m$ is contained in SO_{n+m}. Let $\{e_1, ..., e_{n+m}\}$ be an orthonormal basis of $\mathbb{C}^{n+m}$ such that $SO_n \subset GL(span_{\mathbb{C}}\{e_1, ..., e_n\})$ and $SO_m \subset GL(span_{\mathbb{C}}\{e_{n+1}, ..., e_{n+m}\})$. The connected component of the stabilizer of $e_1 + ie_{n+1}$ in $SO_n \times SO_m$ is $SO_{n-1} \times SO_{m-1}$ (see also [dCP1], Lemma 1.7). Furthermore, $SO(\mathbb{C}^{n+m}) \cdot [e_1 + ie_{n+1}] \subset \mathbb{P}(\mathbb{C}^{n+m})$ is $\mathbb{IG}_1(n+m)$ because $e_1 + ie_{n+1}$ is an anisotropic vector. (Notice that if $R_{G,\theta}$ has type $A_1 \times A_1$, then there is a unique subgroup $G^{\theta} \subset H \subset N_G(G^{\theta})$ such that G/H has a smooth completion with Picard number one). $\square$

3.3 Restricted root system of type G_2

Now, we study the symmetric varieties whose restricted root system has type G_2. First of all, we determine their connected automorphism group; in particular, we prove that such varieties are non-homogeneous.

Lemma 3.2 *Suppose that $R_{G,\theta}$ has type G_2. Then the connected automorphism group $Aut^0(X)$ is isomorphic to G.*

Proof of Lemma 3.2. First, we prove that $Aut^\circ(X)$ does not act transitively over X. This simple variety is associated to the colored cone $(\sigma(\alpha_2^\vee, -\omega_2^\vee), \{D_{\alpha_2^\vee}\})$ and its closed orbit is isomorphic to $G/P(-\omega_1)$. Let ω be the highest weight of Z (with respect to L). The simple restricted root of $R_{[L,L],\theta}$ is α_2, so $(\omega, \alpha_2^\vee) = 1$ (see §3.1). Moreover, $Z(L)^0/(Z(L)^0 \cap H)$ is the one-parameter subgroup of $T/T \cap H$ corresponding to $-\omega_1^\vee$; indeed $(-\omega_1, \omega_1^\vee) < 0$ and $-\omega_1$ belongs to the dual cone $\sigma^\vee$ associated to the closure of $T \cdot x_0$ in Z. Since $Z(L)^0/(Z(L)^0 \cap H) \equiv Z(L)^0 \cdot x_0$ is contained in Z, we have $1 = (\omega, -\omega_1^\vee) = (a\omega_1 + \omega_2, -2\alpha_1 - \alpha_2) = -2a - 1$, so ω is $-\omega_1 + \omega_2$. Thus $Aut^0(X)$ is isomorphic to G by the Remark 3.1. Observe that the connected automorphism group of the closed orbit of $\widetilde{X}$ is G, while the connected automorphism group of the closed orbit of X is either SO_7 or $SO_7 \times SO_7$. □

Proposition 3.5 *Suppose that (G, θ) has type G (and that G/H is isomorphic to $G_2/(SL_2 \times SL_2)$). Then $Aut(X)$ has three orbits in X, is connected and is isomorphic to G_2. Furthermore, X is intersection of hyperplane sections of $\mathbb{G}_3(7)$. More precisely, X is the intersection in $\mathbb{P}(\bigwedge^3 V(\omega_1))$ of $\mathbb{G}_3(7)$ with $\mathbb{P}(V(\omega_1))$. Moreover, X is the subvariety of $\mathbb{G}_3(Im\,\mathbb{O})$ parametrizing the subspaces W such that $\mathbb{C}1 \oplus W$ is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.*

Proof of Proposition 3.5. The center of G_2 is trivial, thus $Aut(X)$ is contained in $Aut_{alg}(G_2)$ (see the observations after the Remark 3.1); such group is connected, so also $Aut(X)$ is connected.

Now, we prove that X is "contained" in $\mathbb{G}_3(7)$. There is an involution of SO_7 that extends θ; again, we denote it by θ. We have $SO_7^\theta = S(O_3 \times SO_4)$, thus $G_2/(SL_2 \times SL_2)$ is a closed subvariety of $SO_7/N_{SO_7}(S(O_3 \times SO_4))$. There is a unique smooth completion of $SO_7/N_{SO_7}(S(O_3 \times SO_4))$ with Picard number one and is isomorphic to $\mathbb{G}_3(7)$ (see Theorem 1.1 and Proposition 4.3). We have to introduce some notations. Let V be a 7-dimensional vector space and let $\{e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3\}$ be a basis of V. Let q be the symmetric bilinear form associated to the quadratic form $(e_0^*)^2 + e_1^* e_{-1}^* + e_2^* e_{-2}^* + e_3^* e_{-3}^*$ and let ϖ be the trilinear form $e_0^* \wedge e_1^* \wedge e_{-1}^* + e_0^* \wedge e_2^* \wedge e_{-2}^* + e_0^* \wedge e_3^* \wedge e_{-3}^* + 2\, e_1^* \wedge e_2^* \wedge e_3^* + 2\, e_{-1}^* \wedge e_{-2}^* \wedge e_{-3}^* \in \bigwedge^3 V^*$. The subgroup G of $SL(V)$ composed by the linear transformations which preserve q and ϖ is the simple group of type G_2; moreover, we can realize SO_7 as $SO(V, q)$. The vector space V is the standard representation $V(\omega_1)$ of G and we can suppose that $\{e_{-3}, e_{-2}, e_1, e_0, e_1, e_2, e_3\}$ is a basis of weight vector for an appropriate maximal torus T of G. Moreover,

we can choose a Borel subgroup B of G so that the weight of e_i is a positive root (respectively is 0) if and only if $i > 0$ (respectively $i = 0$). The Grassmannian $\mathbb{G}_3(7)$ is contained in $\mathbb{P}(\bigwedge^3 V)$ and $\bigwedge^3 V$ is isomorphic to $V \oplus V(2\omega_1) \oplus \mathbb{C}$ as G_2-representation. The subrepresentation of $\bigwedge^3 V$ isomorphic to V has the following basis of T-weights: $\{e_{-2}\wedge e_{-3}\wedge e_0 - e_1\wedge e_2\wedge e_{-2} - e_1\wedge e_3\wedge e_{-3}, e_2\wedge e_{-2}\wedge e_{-3} - e_1\wedge e_2\wedge e_0 + e_1\wedge e_{-1}\wedge e_{-3}, e_3\wedge e_{-2}\wedge e_{-3} - e_1\wedge e_3\wedge e_0 - e_1\wedge e_{-1}\wedge e_{-2}, e_1\wedge e_2\wedge e_3 - e_{-1}\wedge e_{-2}\wedge e_{-3}, e_2\wedge e_3\wedge e_{-3} - e_1\wedge e_2\wedge e_{-1} + e_0\wedge e_{-1}\wedge e_{-3}, -e_2\wedge e_3\wedge e_{-2} - e_1\wedge e_3\wedge e_{-1} - e_0\wedge e_{-1}\wedge e_{-2}, -e_1\wedge e_{-1}\wedge e_{-2} - e_2\wedge e_3\wedge e_0 - e_3\wedge e_{-1}\wedge e_{-3}\}$. Let X'' be the closure of $G_2/(SL_2 \times SL_2)$ in $\mathbb{G}_3(7)$ and let X' be the intersection of $\mathbb{G}_3(7)$ with $\mathbb{P}(V(2\omega_1) \oplus \mathbb{C})$; observe that X'' is contained in X' because $V = V(\omega_1)$ is not spherical (see [T2], Proposition 26.4). We claim that X is the normalization of X''. Indeed $\mathbb{P}(V(2\omega_1) \oplus \mathbb{C})$ has two closed G_2-orbits: one is isomorphic to $G_2/P(\omega_1)$ and the other one is the point $\mathbb{P}(\mathbb{C})$. Therefore X' contains one closed G_2-orbit, otherwise $\mathbb{G}_3(7)$ would contain a G_2-fixed point, in particular V would be reducible as G_2-representation. Thus, the normalization of X'' is a simple variety with closed orbit $G/P(\omega_1)$, so it is X.

We want to prove that X'' coincides with X' and is smooth (so it coincides also with X). Notice that X' is connected, because it is G_2-stable and contains a unique closed G_2-orbit. Hence, it is sufficient to prove that X' has dimension 8 and is smooth in a neighborhood of a point belonging to $G_2/P(\omega_1)$. One can verify that $e_1 \wedge e_{-2} \wedge e_{-3}$ is a highest weight vector of $V(2\omega_2) \subset \bigwedge^3 V$. Let A be the affine open subset of $\mathbb{G}_3(7)$ composed by the subspaces with basis $\{e_j + \sum a_{i,j} e_i\}_{j=1,-2,-3; i=2,3,0,-1}$, where $(a_{i,j})$ varies in $\mathbb{C}^{12}$. The subvariety $A \cap \mathbb{P}(V(2\omega_1 \oplus \mathbb{C})) = A \cap X'$ of A has equations:

$$a_{1,0} = -a_{3,2} + a_{2,3},$$

$$a_{2,-1} = -a_{1,2} + T_{(2,3),(2,0)},$$

$$a_{3,-1} = -a_{1,3} + T_{(2,3),(3,0)},$$

$$a_{1,-1} = T_{(2,3),(2,3)},$$

$$T_{(1,2),(2,3)} - T_{(2,3),(2,-1)} + T_{(1,2),(0,-1)},$$

$$T_{(1,3),(2,3)} - T_{(2,3),(3,-1)} + T_{(1,3),(0,-1)},$$

$$a_{3,-1} + T_{(1,2,3),(2,3,0)} + T_{(1,2),(3,-1)}$$

(where $T_{(h,k),(n,m)}$ is the minor of $(a_{i,j})$ extracted by the h-th and k-th row and by n-th and m-th column). The closed subset A' of A defined by the the first four equation is the graph of a polynomial map, thus it is smooth of dimension 8. Hence the last three equation are identically verified on A', because X'' has dimension 8 (and $A \cap X'' \subset A \cap X' \subset A'$). Therefore $A \cap X'$ coincides with A' and is smooth.

Now, we prove the last statement of the proposition. Identify V with $Im(\mathbb{O})$ and define the associator $[\cdot,\cdot,\cdot] : \bigwedge^3(\mathbb{O}) \to (\mathbb{O})$ as the linear map such that

$[a,b,c] = (ab)c - a(bc)$. This map is G_2-equivariant and has kernel $V(2\omega_1) \oplus \mathbb{C}$. Thus X parametrizes the 3-dimensional subspaces W of $Im(\mathbb{O})$ over which $[\cdot,\cdot,\cdot]$ is zero. Furthermore, $[1,\cdot,\cdot]$ is identically zero. Let W be a subspace associated to a point in X and let $\overline{W}$ be the subalgebra of $\mathbb{O}$ generated by W and 1. It can be either the entire algebra $\mathbb{O}$ or a subalgebra of dimension four. But, if it is the whole algebra, then $\mathbb{O}$ is generated by four elements which associates between them; a contradiction (recall that $\mathbb{O}$ and $\overline{W}$ are composition algebras). Thus, $\overline{W}$ is a composition algebra of dimension four, so it is isomorphic to $\mathbb{H}$. $\square$

Let (V_1, q_1) and (V_2, q_2) be copies respectively of (V, q) and $(V, -q)$, where V is as in the previous proposition; we can suppose $G_2 \times G_2 \subset SO(V_1) \times SO(V_2)$. Moreover, let W_i be a maximal anisotropic subspace of V_i for both the i. Let W be a maximal anisotropic subspace of $V_1 \oplus V_2$ which contains $W_1 \oplus W_2$.

Proposition 3.6 *Suppose that G/H is is isomorphic to the simple group of type G_2. Then $Aut(X)^0$ is $G_2 \times G_2$, while $Aut(X)$ is generated by $Aut(X)^0$ together with the flip $(g,h) \to (h,g)$. In particular, $Aut(X)^0$ has index two in $Aut(X)$. Furthermore, X is intersection of hyperplane sections of $\mathbb{IG}_7(14)$: more precisely, X is the intersection in $\mathbb{P}(\bigwedge^{even}(V_1 \oplus V_2))) \cong \mathbb{P}((V_1 \otimes V_2) \oplus V_1 \oplus V_2 \oplus \mathbb{C})$ of $\mathbb{IG}_7(14)$ with $\mathbb{P}((V_1 \otimes V_2) \oplus \mathbb{C})$*

Proof of Proposition 3.6. The group G has type $G_2 \times G_2$, G^θ is the diagonal and X has dimension 14. The automorphism group of X is determined in [Br4], Example 2.4.5. Alternatively, one can study it in a very similar way to the Proposition 3.5.

Clearly the involution of $G \times G$ can be extended to an involution of $SO(V_1) \times SO(V_2)$ which we denote again by θ; in particular we have $G \cong G \times G/(G \times G)^\theta \subset SO(V_1) \times SO(V_2)/(SO(V_1) \times SO(V_2))^\theta \cong SO(V,q)$.

Let X'' be the closure of G in the unique smooth completion of $SO(V,q)$ with Picard number one. This last variety is isomorphic to the spinorial variety $\mathbb{S}_7$ and the application $\Phi : SO(V,q) \hookrightarrow \mathbb{S}_7$ sends an element $g \in SO(V,q)$ to the graph $graph(g) := \{(v, gv) : v \in V\} \subset V \oplus V \equiv V_1 \oplus V_2$ of g (see also Proposition 4.2). We claim that X is the normalization of X'' (one could show that X'' is normal by [T1], Proposition 9).

Let $\varphi : \mathbb{S}_7 \to \mathbb{P}(\bigwedge^{even} W)$ be the $Spin_{14}$-equivariant embedding of $\mathbb{S}_7$ in the projectivitation of a half-spin representation of $Spin_{14}$. Write $V_i = W_i \oplus \widetilde{W}_i \oplus \mathbb{C}_i$, $V = V_1 \oplus V_2 = W \oplus \widetilde{W}$ where W_i, $\widetilde{W}_i$, $\widetilde{W}$ are maximal anisotropic subspaces such that $\widetilde{W}_1 \oplus \widetilde{W}_2 \subset \widetilde{W}$. The representation $\bigwedge^{even} W$ is isomorphic to $\bigwedge^\bullet W_1 \otimes \bigwedge^\bullet W_2 \cong \bigwedge^\bullet W_1 \otimes (\bigwedge^\bullet W_2)^*$ as $(Spin_7 \times Spin_7)$-representation (see the highest weights and the dimensions). Moreover $\bigwedge^\bullet W_i$ is isomorphic to $V_i \oplus \mathbb{C}_i$ as G_2-representation, so $\bigwedge^\bullet W_1 \otimes \bigwedge^\bullet W_2$ is isomorphic to $(V_1 \otimes V_2) \oplus V_1 \oplus V_2 \oplus \mathbb{C}$ as G-representation. Let $\mathbb{P}$ be the projective subspace of $\mathbb{P}(\bigwedge^{even} W)$ isomorphic to $\mathbb{P}((V_1 \otimes V_2) \oplus \mathbb{C})$. Observe that X'' is contained in $X' := \mathbb{S}_7 \cap \mathbb{P} \subset \mathbb{P}(\bigwedge^{even} W)$ because $V_1 \oplus V_2$ does not contain a line fixed by G^θ (see [T2], Proposition 26.4).

Observe that X' contains one closed G-orbit. Indeed $\mathbb{P}$ contains two closed G-orbits: one isomorphic to $G/P(\omega_1)$ and the other one isomorphic to the G-stable point $\mathbb{P}(\mathbb{C})$. On the other hand, there is not a G-stable maximal isotropic subspace of V, so $\mathbb{P}(\mathbb{C})$ is not contained in $\mathbb{S}_7$. Thus X' is connected. Moreover, the normalization of X'' is the simple symmetric variety with closed orbit $G/P(\omega_1)$, so it is X. We want to prove that X'' is smooth and coincides with X'; it is sufficient to prove that X' is smooth of dimension 12; in this case X' is irreducible, so it coincides with X'' (and X). Moreover, it is sufficient to study X' in a open neighborhood of an arbitrarily fixed point of $G/P(\omega_1)$, for example $x = [e_1 \wedge e_{-2} \wedge e_{-3} \wedge e_0 + f_0 \wedge f_2 \wedge f_3 \wedge f_{-1}]$.

Let $\{e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3\}$ be a basis of V_1 as before and let $\{f_{-3}, f_{-2}, f_1,$ $f_0, f_1, f_2, f_3\}$ be the corresponding basis of V_2. Let u be $e_0 + f_0$; we can suppose that W is generated by $e_1, e_2, e_3, f_1, f_2, f_3$ and u. The trivial subrepresentation $\mathbb{C}_1$ of $\bigwedge^\bullet W_1$ is spanned by $2\sqrt{-2}1_{W_i} + e_1 \wedge e_2 \wedge e_3$; moreover, $W_1 \oplus \bigwedge^2 W_1$ is contained in the G_2-stable subspace of $\bigwedge^\bullet W_1$ isomorphic to V_1. An open neighborhood of x in $\mathbb{S}_7$ is given by $U^- \cdot x$, where U^- is the unipotent radical of standard parabolic subgroup opposite to $Stab_{Spin_{14}}(x)$ (notice that, as algebraic variety, $U^- \cdot x$ is isomorphic to $Lie(U^-)$ by the exponential map). The coordinates of $exp(p) \cdot x$ are the pfaffians of the diagonal minors of p. Let $x_{i,j}$ be the coordinates of the space M_{14} of matrices of order 14 with respect to the basis $\{e_1, e_2, e_3, u, f_1, f_2, f_3,\ e_{-1}, e_{-2}, e_{-3}, \frac{1}{2}(e_0 - f_0), f_{-1}, f_{-2}, f_{-3}\}$ (notice that $Lie(Spin_{14}) \subset M_{14}$). We claim that a open neighborhood of x in $X' \cap (U^- \cdot x)$ is the graph of a polynomial map. Given an skew-symmetric matrix of order $2n$, let $[i_1, i_2, ..., i_{2k}]$ be the Pfaffian of the principal minor extracted from the rows and the columns of indices $i_1 < i_2 < ... < i_{2k}$.

The three equations corresponding to the vectors $2\sqrt{-2}e_i \wedge e_j + e_i \wedge e_j \wedge u \wedge f_1 \wedge f_2 \wedge f_3$ ($\in V_1$) are, respectively, $0 = x_{i,j} - [i,j,4,5,6,7] = x_{i,j} - (x_{4,7}p_{i,j} + q_{i,j})$, where the $p_{i,j}$, $q_{i,j}$ are homogeneous polynomials in the $x_{h,k}$ such that : 1) $(h,k) \neq (4,7)$; 2) either $h > 3$ or $k > 3$. Finally, we consider the equation $0 = x_{4,7} - [1,2,3,4] = x_{4,7} - (x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{2,3}x_{1,4})$ associated to $f_4 \wedge u + f_4 \wedge e_1 \wedge e_2 \wedge e_3$ ($\in V_2$). Substituting the first three equation in the last one, we obtain $x_{4,7}(1 + x_{1,2}f_{1,2} + x_{1,3}f_{1,3} + x_{2,3}f_{2,3}) = g$ where the $f_{i,j}$ and g are polynomial in the $x_{4,j}$ with $j < 4$.

Therefore, there is an open neighborhood A of x in $U^- \cdot x$ where the previous four equation became $x_{1,2} = h_{1,2}$, $x_{1,3} = h_{1,3}$, $x_{2,3} = h_{2,3}$ and $x_{4,7} = h_{4,7}$ (here the $h_{i,j}$ are polynomials in the coordinates different from $x_{1,2}$, $x_{1,3}$, $x_{2,3}$ and $x_{4,7}$). Observe that the previous equation are independent; on the other hand X' has dimension at least 12, because it contains X''. Therefore, the subvariety A' of A obtained imposing the previous four equation is smooth with dimension 12, thus it is equal to $A \cap X''$. Hence $X' = X''(= X)$. $\square$

3.4 Restricted root system of type A_2 (with $H = G^\theta$)

Now, we consider the symmetric varieties such that: 1) the restricted root system has type A_2; 2) $H = G^\theta$. We prove that they are hyperplane sections of the varieties of the subexceptional serie of subadjoint varieties (these last varieties constitute the third line of the geometric version of Freudenthal's magic square and are Legendrian varieties). For the completion of SL_2 this result is due to Buczyński (see [Bu]). These varieties are contained nested in each other. First, we prove that such varieties are hyperplane sections of Legendrian varieties. Then, we study their connected automorphism group; in particular we show that these varieties are not homogeneous. Finally, we study their automorphism group.

Recall that $\mathcal{J}_3(\mathbb{A})$ is the space of Hermitian matrices of order three over the complex composition algebra $\mathbb{A}$. Moreover $SL_3(\mathbb{A})$ is the subgroup of $GL_{\mathbb{C}}(\mathcal{J}_3(\mathbb{A}))$ of complex linear transformations preserving the determinant, while $SO_3(\mathbb{A})$ is the subgroup of complex linear transformations preserving the Jordan multiplication. The space $\mathcal{Z}_2(\mathbb{A}) := \mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})^* \oplus \mathbb{C}^*$ is an irreducible $Sp_6(\mathbb{A})$-representation. The closed $Sp_6(\mathbb{A})$-orbit in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$ is $\mathbb{LG}(\mathbb{A}^3, \mathbb{A}^6)$ and is the image of the $Sp_6(\mathbb{A})$-equivariant rational map:

$$\begin{array}{rcc} \phi : \mathbb{P}(\mathbb{C} \oplus \mathcal{J}_3(\mathbb{A})) & \dashrightarrow & \mathbb{P}(\mathcal{Z}_2(\mathbb{A})) \\ (x, P) & \rightarrow & (x^3, x^2 P, x com\, P, det\, P). \end{array}$$

The quotient $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ is a symmetric variety isomorphic to the image of $SL_3(\mathbb{A}) \cdot [1, I]$ by ϕ; in particular it is contained in the hyperplane section $X' := \{[x_1, x_2, x_3, x_4] \in \mathbb{LG}(\mathbb{A}^3, \mathbb{A}^6) \subset \mathbb{P}(\mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})^* \oplus \mathbb{C}^*) :\ x_1 = x_4\}$. Such section is irreducible because it is the image by ϕ of $\{[x, P] : x^3 - det(P)\}$; moreover, it has the same dimension of $SL_3(\mathbb{A})/SO_3(\mathbb{A})$, thus it is the closure of $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$.

Lemma 3.3 *The variety X' is isomorphic to the unique smooth completion X of $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ with Picard number one.*

Proof of Lemma 3.3. The variety X' and its normalization X'' have the same number of orbits by [T1], Proposition 1; moreover, each orbit of X'' have the same dimension of its image in X'. Observe that no orbits of X'' have dimension 0; thus X' has two closed orbits, namely $G/P(\omega_1)$ and $G/P(\omega_2)$. One can easily show that X'' has a unique orbit $\mathcal{O}$ of codimension one (otherwise, by the theory of spherical embeddings, X' would contain a closed orbit isomorphic to $G/P(\omega_1 + \omega_2)$). Thus $X'' = G/H \cup \mathcal{O} \cup G/P(\omega_1) \cup G/P(\omega_2)$. By Proposition 8.2 in [LM], we know the possible dimensions for the singular locus of X'. Because it is $SL_3(\mathbb{A})$-stable, one can easily see that X' is smooth (and coincides with X''). Studying the colored fan of X'', one can show easily that X'' has Picard number one, so it is isomorphic to X. $\square$

Observe that we have proved the following proposition.

Proposition 3.7 *The smooth completion of $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ with Picard number one is contained in the smooth completion of $SL_3(\mathbb{A}')/SO_3(\mathbb{A}')$ with Picard number one and $\mathbb{A}' \subset \mathbb{A}$. Moreover, we have a commutative diagram:*

$$\begin{tikzcd}
\overline{SL_3/SO_3} \arrow[r, hook] \arrow[d, hook] & \overline{SL_3} \arrow[r, hook] \arrow[d, hook] & \overline{SL_6/Sp_6} \arrow[r, hook] \arrow[d, hook] & \overline{E_6/F_4} \arrow[d, hook] \\
\mathbb{LG}_3(6) \arrow[r, hook] & \mathbb{G}_3(6) \arrow[r, hook] & \mathbb{S}_{12} \arrow[r, hook] & \mathbb{G}_3(\mathbb{O}^6)
\end{tikzcd}$$

Lemma 3.4 *Suppose that $R_{G,\theta}$ has type A_2 and that $H = G^\theta$. Then the connected automorphism group $Aut^0(X)$ is isomorphic to G, up to isogeny.*

Proof of Lemma 3.4. The character group of $\chi(S)$ is the lattice generated by spherical weights. Let X' be the simple variety corresponding to the colored cone $(\sigma(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_1^\vee}\})$; its closed orbit is isomorphic to $G/P(\omega_2)$. Let Z be as §3.1 and let $\omega = a_1\omega_1 + a_2\omega_2$ be the highest weight of Z (with respect to L). The simple restricted root of $R_{[L,L],\theta}$ is α_1, so $a_1 = (\omega, \alpha_1^\vee) = 1$ (see §3.1). Moreover, $Z(L)^0/(Z(L)^0 \cap H)$ is the one-parameter subgroup of $T/T \cap H$ corresponding to $-3\omega_2^\vee$; indeed $(-\omega_2, \omega_2^\vee) < 0$ and $-\omega_2$ belongs to the dual cone $\sigma^\vee$ associated to the closure of $T \cdot x_0$ in Z. Since $Z(L)^0/(Z(L)^0 \cap H) \equiv Z(L)^0 \cdot x_0$ is contained in Z, we have $1 = (\omega, -3\omega_2^\vee) = -1 - 2a_2$, so ω is $\omega_1 - \omega_2$. We can study he simple variety corresponding to the colored cone $(\sigma(\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_2^\vee}\})$ in a similar way. Thus $Aut^0(X)$ is isomorphic to G by the Remark 3.1. □

Proposition 3.8 *Suppose that G/H is SL_3/SO_3 (and the type of (G,θ) is AI). Then X is an hyperplane section of $\mathbb{LG}_3(6)$. The full automorphism group $Aut(X)$ is generated by SL_3 ($= Aut^0(X)$) and by θ; moreover $Aut(X)$ has three orbits in X.*

Proposition 3.9 *(see [Bu], Theorem 1.4) Suppose that G/H is SL_3. Then X is an hyperplane section of $\mathbb{G}_3(6)$. Moreover $Aut^0(X)$ is isomorphic to the quotient of $SL_3 \times SL_3$ by the intersection of the center with the diagonal. The full automorphism group $Aut(X)$ is generated by $Aut^0(X)$, by θ and by (φ, φ) where φ is the automorphism of SL_3 corresponding to the automorphism of the Dynkin diagram; in particular $Aut^0(X)$ has index 4 in $Aut(X)$. Moreover $Aut(X)$ has three orbits in X.*

Proposition 3.10 *Suppose that G/H is SL_6/Sp_6 (and the involution's type is AII). Then X is an hyperplane section of $\mathbb{S}_{12}$. The full automorphism group $Aut(X)$ is generated by $SL_6/\{\pm id\}$ ($= Aut^0(X)$) and by θ; thus $Aut^0(X)$ has index 2 in $Aut(X)$. Moreover $Aut(X)$ has three orbits in X.*

Proposition 3.11 *Suppose that G/H is E_6/F_4 (and the involution's type is AII). Then X is an hyperplane section of $E_7/P_7 \equiv \mathbb{LG}(\mathbb{A}^3, \mathbb{A}^6)$. Moreover $Aut^0(X)$ is isomorphic to the simply-connected group of type E_6, while the full automorphism group $Aut(X)$ is generated by E_6 and by θ; in particular $Aut^0(X)$ has index 2 in $Aut(X)$. Besides $Aut(X)$ has three orbits in X.*

Now, we study the automorphism group of X. By the Lemma 3.4, we know that $Lie(Aut(X))$ is $Lie(G)$. Notice that the center of G acts no-trivially over X. Moreover, if G is different from $SL_3 \times SL_3$ and SL_6, then the center of G is a simple group, thus $Aut^\circ(X)$ is G. If G is $SL_3 \times SL_3$, then the center is $C_3 \times C_3$, where C_n is the group of n-th roots of the unit. Thus the intersection of the center of G with G^θ is the diagonal of $C_3 \times C_3$. If G is SL_6 then the center is C_6 and its intersection with $Sp_6(= G^\theta)$ is $\{\pm id\}$. Observe that $N_G(H)/H$ is simple because the fundamental group of $R_{G,\theta}$ is A_2.

Now, we determine K (see §3.1). It sufficient to determine the subset K' of E. Let ϖ be the longest element of W and let ϖ_0 be the longest element of the Weyl group of R^0_G (here we consider the root system of G with respect to T). Then $\theta' := \omega\omega_0\theta$ fixes T and B. One can easily show that θ' exchange ω_1 with ω_2 (see [T2], table 5.9 and [dCP1], §1.4), thus it exchanges E_1 with E_2 (recall that the restriction of the valuation of E_i to $\mathbb{C}(X)^{(B)}/\mathbb{C}^*$ is $-\omega_i$). Hence, also θ exchanges the previous two divisors; in particular θ exchanges the two closed orbits of $\widetilde{X}$. Notice that θ belongs to K' and induces the non-trivial automorphism of the Dynkin diagram of G, with respect to T' and B' (see page 493 in [LM] and §26 in [T2]). If G is different from $SL_3 \times SL_3$, then $E/\overline{T}'$ contains exactly two elements. Thus $K'/N_{T'}(H)$ coincides with $E/\overline{T}'$. In particular, $Aut(G/H)$ is generated by $Aut^0(G/H)$ and θ. Moreover, $Aut(G/H)$ coincides with $Aut(X)$ by Remark 3.1.

Finally, suppose $G = SL_3 \times SL_3$. Let $\dot{T}$ be a maximal torus of SL_3, let $\dot{B}$ be a Borel subgroup of SL_3 and let φ be the equivariant automorphism of SL_3 associated to the non-trivial automorphism of the Dynkin diagram (with respect to $\dot{T}$ and $\dot{B}$). We can set $T = T' = \dot{T} \times \dot{T}$, $B = \dot{B} \times \dot{B}^-$ and $B' = \dot{B} \times \dot{B}$. One can easily see that K' is generated by $N_{T'}(H)$, θ and (φ, φ). (Notice that $E/\overline{T}'$ has eight elements: id, θ, (φ, id), (id, φ), (φ, φ), $\theta \circ (\varphi, \varphi)$, $\theta \circ (\varphi, id)$ and $\theta \circ (id, \varphi)$). Notice that (φ, φ) stabilizes both B and B^-. We have $\omega_i = \varpi_i^1 - \varpi_i^2$ where $\{\varpi_1^j, \varpi_2^j\}$ are the fundamental weights of the j-th copy of SL_3 in G (with respect to $\dot{T}$ and $\dot{B}$). Thus (φ, φ) exchange ω_1 with ω_2, so (φ, φ) exchange E_1 with E_2. Furthermore, (φ, φ) induces a non-trivial automorphism of the Dynkin diagram of G (with respect to T' and B'). Therefore, $Aut(X)$ coincides with $Aut(G/H)$ by Remark 3.1.

4 Varieties of rank at least three

In the following we always suppose that the rank of G/H is at least two. In this section we consider the symmetric varieties which belongs to an infinite family; in particular, we consider all the symmetric varieties of rank at least three. We consider also completions of the following varieties of rank two: 1) the symmetric variety PSL_3; 2) the symmetric variety PSL_3/PSO_3 of type AI; 3) the symmetric variety PSL_4/PSp_4 of type AII; 4) the symmetric variety SO_5; 5) the symmetric variety $Spin_5 \equiv Sp_4$; 6) the symmet-

ric varieties $SO_n/N_{SO_n}(SO_2 \times SO_{m-2})$ of type BI and DI; 7) the symmetric varieties $Sp_{2n}/N_{Sp_{2n}}(Sp_2 \times Sp_{2n-2})$ of type CII; 8) the symmetric variety $SO_8/N_{SO_8}(GL_4)$ of type $DIII$.

Given a linear endomorphism φ of a vector space V, let $graph(\varphi)$ be the subspace $\{(v, \varphi(v)) : v \in V\}$ of $V \oplus V$. Observe that $graph(\varphi)$ has the same dimension of V. If we have fixed a (skew-)symmetric bilinear form on V, then we define a (skew-)symmetric bilinear q' form on $V \oplus V$ such that $q'(v, w) = q(v) - q(w)$ for each $(v, w) \in V \oplus V$.

4.1 Restricted root system of type A_l

Now, we consider the symmetric varieties such that: 1) $H = N_G(G^\theta)$; 2) the restricted root system has type A_l (with $l \geq 2$).

Proposition 4.1 *Let X be a smooth projective symmetric variety with Picard number one and rank at least two. Suppose that $H = N_G(G^\theta)$ and that the restricted root system has type A_l, then X is isomorphic to the projectivization of an irreducible G-representation.*

Proof. It is sufficient to show an irreducible spherical representation with dimension equal to dim $G/H + 1$. We have already considered the case of $E_6/N_{E_6}(F_4)$ in Proposition 3.2.

1) If G/H is isomorphic to PGL_{l+1}, then X is isomorphic to $\mathbb{P}(M_{l+1})$ as $(PGL_{l+1} \times PGL_{l+1})$-variety (here M_{l+1} is the space of matrices of order $l+1$). Indeed $PGL_{l+1} \times PGL_{l+1}$ acts on M_{l+1} and the stabilizer of $\mathbb{C}\mathit{1}$ is the diagonal, namely G^θ. Moreover $\mathbb{P}(M_{l+1})$ has dimension equal to $l^2 + l$.

2) If G/H is isomorphic to $SL_{l+1}/N_{SL_{l+1}}(SO_{l+1})$ and (G, θ) has type AI, then X is isomorphic to $\mathbb{P}(Sym^2(\mathbb{C}^{l+1}))$ (see [CM], Theorem 2.3 or [T2], Proposition 26.4).

3) If G/H is isomorphic to $SL_{l+1}/N_{SL_{l+1}}(Sp_{l+1})$ and (G, θ) has type AII, then X is isomorphic to $\mathbb{P}(\bigwedge^{2l-2}(\mathbb{C}^{2l}))$. $\square$

4.2 Restricted root system of type B_l or D_l

Now we consider the symmetric varieties whose restricted root system has type D_l or B_l. In the second case we suppose also $H = N_G(G^\theta)$. Explicitly, we consider the following cases: 1) the symmetric varieties SO_n; 2) the symmetric varieties $SO_n/(SO_l \times SO_{n-l})$ of type BI and DI. We consider also the symmetric variety SO_8/GL_4 of type $DIII$ because it is isomorphic to $SO_8/(SO_2 \times SO_6)$.

Proposition 4.2 *The smooth projective embedding of SO_n with Picard number one is isomorphic to $\mathbb{S}_n$.*

Proof. We have an inclusion of SO_n into $\mathbb{IG}_n(2n)$ given by the map $g \to graph(\varphi)$. It is easy to show that this map is compatible with the action of $SO_n \times SO_n$ on SO_n and $\mathbb{IG}_n(2n)$. Thus it is sufficient to observe that SO_n and $\mathbb{IG}_n(2n)$ have the same dimension. $\square$

Proposition 4.3 *The smooth projective embedding of $SO_n/(SO_l \times SO_{n-l})$ with Picard number one is isomorphic to $\mathbb{G}_l(n)$.*

Proof. Let W be a l-dimensional subspace of $\mathbb{C}^n$ over which the quadratic form is nondegenerate. We denote by $W^\perp$ the orthogonal complement of W. The stabilizer of W contains $SO(W) \times SO(W^\perp)$. Thus $\mathbb{G}_l(n)$ is a smooth completion with Picard number one of the symmetric variety $SO_n/Stab_{SO_n}(W)$ (see [dCP1], Lemma 1.7). This fact implies, by Theorem 1.1, that H is $SO(W) \times SO(W^\perp)$ (notice that G/H cannot be $Spin_n/(Spin_2 \times Spin_{n-2})$ because this last variety is Hermitian and its restricted root system has type B_2). □

Proposition 4.4 *The smooth projective embedding of SO_8/GL_4 with Picard number one is isomorphic to $\mathbb{G}_2(8)$.*

Proof. Observe that SO_8/GL_4 is isomorphic to $SO_8/S(O_2 \times O_6)$. Indeed the involution of SO_8 of type $DIII$ (and rank two) is conjugated by an equivariant automorphism of SO_8 to the involution of SO_8 of type DI and (and rank two). Indeed, these varieties have the same Satake diagrams up to an automorphism of the Dynkin diagram of SO_8. Now, it is sufficient to observe that a homogeneous symmetric variety has at most one smooth completion with Picard number one. □

4.3 Restricted root system with type BC_l or C_l

Now, we consider the symmetric varieties whose restricted root system has type BC_l or C_l. We have the following two cases: 1) the symmetric varieties $Sp_{2n}/(Sp_{2l} \times Sp_{2n-2l})$ of type CII; 2) the symmetric varieties Sp_{2l}. We consider also $Spin_5$, because it is isomorphic to Sp_4.

Proposition 4.5 *The smooth projective embedding of $Sp_{2n}/Sp_{2l} \times Sp_{2n-2l}$ with Picard number one is isomorphic to $\mathbb{G}_{2l}(2n)$.*

Proof. Let W be a $2l$-dimensional subspace of $\mathbb{C}^{2n}$ over which the standard bilinear skew-symmetric form is nondegenerate and let $W^\perp$ be its orthogonal complement. The Sp_{2n}-orbit of W in $\mathbb{G}_{2l}(2n)$ is isomorphic to $Sp_{2n}/Sp(W) \times Sp(W^\perp)$ (see [dCP1], Lemma 1.7), thus is dense in $\mathbb{G}_{2l}(2n)$. □

Proposition 4.6 *The smooth projective embedding of $Sp(2l)$ with Picard number one is isomorphic to $\mathbb{IG}_{2l}(4l)$.*

Proof. We have an inclusion of $Sp(2l)$ into $\mathbb{IG}_{2l}(4l)$ given by the map $g \to graph(\varphi)$. It is easy to show that this map is compatible with the action of $Sp(2l) \times Sp(2l)$ over $Sp(2l)$ and $\mathbb{IG}_{2l}(4l)$. Furthermore, $Sp(2l)$ and $\mathbb{IG}_{2l}(4l)$ have the same dimension. □

Proposition 4.7 *The smooth projective embedding of $Spin_5$ with Picard number one is isomorphic to $\mathbb{LG}_4(8)$.*

Proof. Observe that $Spin_5 \times Spin_5$ is isomorphic to $Sp_4 \times Sp_4$. □

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